Journal of Statistical Physics, Vol. 124, Nos. 2-4, July/August 2006 (© 2006)
DOI: 10.1007/s10955-006-9025-y

Self-Similarity and Power-Like Tails in Nonconservative Kinetic Models

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Received April 19, 2005; accepted January 6, 2006
Published Online: March 28, 2006

In this paper, we discuss the large-time behavior of solution of a simple kinetic model of Boltzmann–Maxwell type, such that the temperature is time decreasing and/or time increasing. We show that, under the combined effects of the nonlinearity and of the time-monotonicity of the temperature, the kinetic model has non trivial quasi-stationary states with power law tails. In order to do this we consider a suitable asymptotic limit of the model yielding a Fokker-Planck equation for the distribution. The same idea is applied to investigate the large-time behavior of an elementary kinetic model of economy involving both exchanges between agents and increasing and/or decreasing of the mean wealth. In this last case, the large-time behavior of the solution shows a Pareto power law tail. Numerical results confirm the previous analysis.

KEY WORDS: Granular gases, overpopulated tails, Boltzmann equation, wealth and income distributions, Pareto distribution.

1. INTRODUCTION

A well-known phenomenon in the large-time behavior of the Boltzmann equation with dissipative interactions is the formation of overpopulated tails.(3,14,15) Exact results on the behavior of these tails have been obtained for simplified models, in particular for a gas of inelastic Maxwell particles. Our goal here is to show that, at least for some simplified kinetic model, the formation of overpopulated tails is not only a behavior typical of systems where there is dissipation of the temperature (cooling), but more generally is a consequence of the fact that the temperature is not conserved. One can indeed conjecture that the formation

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of overpopulated tails in a kinetic model depends on the breaking of energy conservation. In kinetic theory of rarefied gases, formation of overpopulated tails has been first observed for inelastic Maxwell models.[14,15] Inelastic Maxwell models share with elastic Maxwell molecules the property that the collision rate in the Boltzmann equation is independent of the relative velocity of the colliding pair. These models are of interest for granular fluids in spatially homogeneous states because of the mathematical simplifications resulting from a velocity independent collision rate. Among others properties, the inelastic Maxwell models exhibit similarity solutions, which represent the intermediate asymptotic of a wide class of initial conditions.[7] Recently, the study of a dissipative kinetic model obtained by generalizing the classical model known as Kac caricature of a Maxwell gas,[31] led to new ideas on the mechanism of the formation of tails. Indeed, in[31] connections between the cooling problem for the dissipative model and the classical central limit theorem for stable laws of probability theory were found. A second point in favor of our conjecture on tails formation comes out from some recent applications to economy of one-dimensional kinetic models of Maxwell type.[13,26,32] The main physical law here is that a strong economy produces growth of the mean wealth (which of course is the opposite phenomenon to the dissipation). Nevertheless, the kinetic model led to an immediate explanation of the formation of Pareto tails.[28] Having this in mind, in the next Section we study a one-dimensional Boltzmann-like equation which is able to describe both dissipation and production of energy. This model has been recently considered in[2] with the aim of recovering exact self-similar solutions. The analysis of,[2] based on the possibility to use Fourier transform techniques to investigate properties of the self-similar profiles, shows that in many cases there is evidence of algebraic decay of the velocity distributions. On the other hand, except in particular cases, no exact results can be achieved. To obtain a almost complete description of the large time behavior of the solution, we resort to a different approach. After a brief description of the model, in Sec. 2 we introduce a suitable asymptotic analysis, which reduces the Boltzmann equation to a Fokker-Planck like equation which has an explicitly computable stationary state with power-like tails. In Sec. 3, we show how similar ideas can be fruitfully applied to describe the large-time behavior of some elementary kinetic models of an open economy. Here, the underlying Fokker-Planck equation takes the form of a similar one introduced recently in.[9,13] The rest of the paper is devoted to the proof of mathematical details. Numerical experiments on the Boltzmann models can be found at the end of the paper.

2. KINETIC MODELS AND FOKKER-PLANCK ASYMPTOTICS

In this section we will study the large-time behavior of solutions to one-dimensional kinetic models of Maxwell-Boltzmann type, where the binary interaction between particles obey to the law

$$v^* = pv + qw, \quad w^* = qv + pw; \quad p > q > 0. \tag{1}$$

The positive constants p and q represent the interacting parameters, namely the portion of the pre-collisional velocities (v, w) which generate the post-collisional ones (v^*, w^*). As it will be clear after Subsection 2.2, the choice $p > q$ is natural in mimicking economic interactions, so that we will assume it even in molecular dynamics. As a matter of fact, the mixing parameters p and q can be exchanged, which corresponds to the exchange of post-collision velocities, without any change in the global collision evolution.

2.1. Nonconservative Kinetic Models

Let $f(v, t)$ denote the distribution of particles with velocity $v \in I\!R$ at time $t \geq 0$. The kinetic model can be easily derived by standard methods of kinetic theory, considering that the change in time of $f(v, t)$ depends on a balance between the gain and loss of particles with velocity v due to binary collisions. This leads to the following integro-differential equation of Boltzmann type,[2]

$$\frac{\partial f}{\partial t} = \int_{I\!R} \left(\frac{1}{J} f(v_*) f(w_*) - f(v) f(w) \right) dw \tag{2}$$

where (w_*, w_*) are the pre-collisional velocities that generate the couple (v, w) after the interaction. In (2) $J = p^2 - q^2$ is the Jacobian of the transformation of (v, w) into (v^*, w^*). Note that, since we fixed $p > q$, the Jacobian J is positive and that the unique situation corresponding to $J = 1$ is obtained taking $p = 1$ and $q = 0$ for which the collision operator vanishes.

The kinetic Eq. (2) is the analogous of the Boltzmann equation for Maxwell molecules,[4,12] where the collision frequency is assumed to be constant. Also, it presents several similarities with the one-dimensional Kac model.[21,24] It is well-known to people working in kinetic theory that this simplification allows for a better understanding of the qualitative behavior of the solutions.

Without loss of generality, we can fix the initial density to satisfy

$$\int_{I\!R} f_0(v)\, dv = 1\,; \quad \int_{I\!R} v f_0(v)\, dv = 0 \quad \int_{I\!R} v^2 f_0(v)\, dv = 1. \tag{3}$$

To avoid the presence of the Jacobian, and to study approximation to the collision operator it is extremely convenient to write Eq. (2) in weak form. It corresponds to consider, for all smooth functions $\phi(v)$, the equation

$$\frac{d}{dt} \int_{I\!R} \phi(v) f(v, t)\, dv = \int_{I\!R^2} f(v) f(w) (\phi(v^*) - \phi(v))\, dv\, dw. \tag{4}$$

One can alternatively use the symmetric form

$$\frac{d}{dt}\int_{I\!R} f(v)\phi(v)\,dv = \frac{1}{2}\int_{I\!R^2} f(v)f(w) \\ (\phi(v^*) + \phi(w^*) - \phi(v) - \phi(w)\,dv\,dw. \qquad (5)$$

A remarkable fact is that Eqs. (4) and (5) can be studied for all values of the mixing parameters p and q, including the case $p = q$, which could not be considered in Eq. (3).

Choosing $\phi(v) = v$, (respectively $\phi(v) = v^2$) shows that

$$m(t) = \int_{I\!R} vf(v,t)\,dv = m(0)\exp\{(p + q - 1)t\}. \qquad (6)$$

Hence, since the initial density f_0 satisfies (3), $m(0) = 0$ and $m(t) = 0$ for all $t > 0$. Consequently,

$$E(t) = \int_{I\!R} v^2 f(v,t)\,dv = \exp\{(p^2 + q^2 - 1)t\}. \qquad (7)$$

Higher order moments can be evaluated recursively, remarking that the integrals $\int v^n f(v,t)$ obey a closed hierarchy of Eq. (3).

Note that the second moment of the solution is not conserved, unless the collision parameters satisfy

$$p^2 + q^2 = 1.$$

If this is not the case, the energy can grow to infinity or decrease to zero, depending on the sign of $p^2 + q^2 - 1$. In both cases, however, stationary solutions of finite energy do not exist, and the large-time behavior of the system can at best be described by self-similar solutions. The standard way to look for self-similarity is to scale the solution according to the role

$$g(v,t) = \sqrt{E(t)}\,f\big(v\sqrt{E(t)},t\big). \qquad (8)$$

This scaling implies that $\int v^2 g(v,t) = 1$ for all $t \geq 0$. Elementary computations show that $g = g(v,t)$ satisfies the equation

$$\frac{\partial g}{\partial t} - \frac{1}{2}(p^2 + q^2 - 1)\frac{\partial}{\partial v}(vg) = \int_{I\!R}\left(\frac{1}{J}g(v_*)g(w_*) - g(v)g(w)\right)dw. \qquad (9)$$

In weak form, Eq. (9) reads

$$\frac{d}{dt}\int_{I\!R}\phi(v)g(v,t)\,dv \ -\frac{1}{2}(p^2 + q^2 - 1)\int_{I\!R}\phi(v)\frac{\partial}{\partial v}(vg)\,dv \\ = \int_{I\!R^2} g(v)g(w)(\phi(v^*) - \phi(v))\,dv\,dw. \qquad (10)$$

Assuming that ϕ vanishes at infinity, we can integrate by parts the second integral on the right-hand side of (10) to obtain

$$\frac{d}{dt}\int_{I\!R}\phi(v)g(v,t)\,dv \;+\; \frac{1}{2}(p^2+q^2-1)\int_{I\!R}\phi'(v)vg(v)\,dv \tag{11}$$
$$= \int_{I\!R^2} g(v)g(w)(\phi(v^*)-\phi(v))\,dv\,dw.$$

By the collision rule (1),

$$v^* - v = (p-1)v + qw.$$

Let us use a second order Taylor expansion of $\phi(v^*)$ around v

$$\phi(v^*)-\phi(v) = ((p-1)v+qw)\,\phi'(v) + \frac{1}{2}\,((p-1)v+qw)^2\,\phi''(\tilde{v}),$$

where, for some $0 \le \theta \le 1$

$$\tilde{v} = \theta v^* + (1-\theta)v.$$

Inserting this expansion in the collision operator, we obtain the equality

$$\int_{I\!R^2} g(v)g(w)(\phi(v^*) \;-\phi(v))\,dvdw = \int_{I\!R^2} g(v)g(w)((p-1)v$$
$$+\,qw)\,\phi'(v)\,dvdw + \frac{1}{2}\int_{I\!R^2} g(v)g(w)((p-1)v$$
$$+\,qw)^2\phi''(v)\,dvdw + R(p,q), \tag{12}$$

where

$$R(p,q) = \frac{1}{2}\int_{I\!R^2} ((p-1)v+qw)^2\,(\phi''(\tilde{v})-\phi''(v))g(v)g(w)\,dv\,dw. \tag{13}$$

Recalling that $g(v,t)$ satisfies (3), we can simplify into (12) to obtain

$$\int_{I\!R^2} g(v)g(w)(\phi(v^*)-\phi(v))\,dv\,dw = (p-1)\int_{I\!R} vg(v)\phi'(v)\,dv$$
$$\frac{1}{2}\int_{I\!R} g(v)((p-1)^2v^2+q^2)\phi''(v)\,dv + R(p,q). \tag{14}$$

Substituting (14) into (11), and grouping similar terms, we conclude that $g(v,t)$ satisfies

$$\frac{d}{dt}\int_{I\!R}\phi(v)g(v,t)\,dv + \frac{1}{2}\left((p-1)^2+q^2\right)\int_{I\!R}\phi'(v)vg(v)\,dv$$
$$= \frac{1}{2}\int_{I\!R} g(v)\left((p-1)^2v^2+q^2\right)\phi''(v)dv + R(p,q). \tag{15}$$

Hence, if we set

$$\tau = q^2 t, \quad h(v,\tau) = g(v,t), \tag{16}$$

which implies $g_0(v) = h_0(v)$, $h(v,\tau)$ satisfies

$$\begin{aligned}\frac{d}{d\tau}\int_{I\!R}\phi(v)h(v,\tau)\,dv &+ \frac{1}{2}\left(\left(\frac{p-1}{q}\right)^2+1\right)\int_{I\!R}\phi'(v)vh(v)\,dv\\ &= \frac{1}{2}\int_{I\!R}h(v)\left(\left(\frac{p-1}{q}\right)^2 v^2+1\right)\phi''(v)\,dv + \frac{1}{q^2}R(p,q).\end{aligned} \tag{17}$$

Suppose now that the remainder in (17) is small for small values of the parameter q. Then Eq. (17) gives the behavior of $g(v,t)$ for large values of time. Moreover, taking $p = p(q)$ such that, for a given constant λ

$$\lim_{q\to 0}\frac{p(q)-1}{q} = \lambda, \tag{18}$$

Equation (17) is well-approximated by the equation (in weak form)

$$\begin{aligned}\frac{d}{d\tau}\int_{I\!R}\phi(v)h(v,\tau)\,dv &+ \frac{1}{2}(\lambda^2+1)\int_{I\!R}\phi'(v)vh(v)\,dv\\ &= \frac{1}{2}\int_{I\!R}h(v)\left(\lambda^2v^2+1\right)\phi''(v)\,dv.\end{aligned} \tag{19}$$

Equation (19) is nothing but the weak form of the Fokker-Planck Equation.

$$\frac{\partial h}{\partial \tau} = \frac{1}{2}\left(\frac{\partial^2}{\partial v^2}((1+\lambda^2v^2)h) + (1+\lambda^2)\frac{\partial}{\partial v}(vh)\right), \tag{20}$$

which has a unique stationary state of unit mass, given by

$$M_\lambda(v) = c_\lambda\left(\frac{1}{1+\lambda^2v^2}\right)^{\frac{3}{2}+\frac{1}{2\lambda^2}}, \tag{21}$$

where

$$c_\lambda = \frac{|\lambda|}{\sqrt{\pi}}\frac{\Gamma\left(\dfrac{3\lambda^2+1}{2\lambda^2}\right)}{\Gamma\left(\dfrac{1+2\lambda^2}{2\lambda^2}\right)}. \tag{22}$$

Remark 2.1. *The derivation of the Fokker-Planck equation (20) presented in this section is largely formal. The main objective here was to show that there are regimes of the mixing parameters for which we can expect formation of self-similar* *solutions to the kinetic model with overpopulated tails. We postpone the detailed proof and the mathematical technicalities to the second part of the paper.*

Remark 2.2. *The conservative case $p^2 + q^2 = 1$ can be treated likewise. In this case one is forced to choose $p = \sqrt{1 - q^2}$, which gives $\lambda = 0$ as unique possible value. In the limit one then obtains the linear Fokker-Planck equation*

$$\frac{\partial h}{\partial \tau} = \frac{1}{2} \left(\frac{\partial^2 h}{\partial v^2} + \frac{\partial}{\partial v} (vh) \right). \tag{23}$$

Note that in this case the stationary solution $M(v)$ is the Maxwell density

$$M(v) = \frac{1}{\sqrt{2\pi}} e^{-v^2/2}, \tag{24}$$

for all $q < 1/\sqrt{2}$. On the contrary, the non conservative cases are characterized by a λ different from zero, which produces a stationary state with overpopulated tails. Note that from (22) we have $c_\lambda \to 1/\sqrt{2\pi}$ as $\lambda \to 0$ and thus $M(v) = \lim_{\lambda \to 0} M_\lambda(v)$.

Remark 2.3. *The possibility to pass to the limit in (18), with $\lambda > 0$, is restricted to the cases $p^2 + q^2 < 1$ and $p^2 + q^2 > 1$, but $p > 1$. In the case $p^2 + q^2 > 1$, $p < 1$, it holds*

$$0 < \frac{(1-p)^2}{q^2} < \frac{1-p}{1+p},$$

which forces λ towards zero as $p \to 1$. This case, as the conservative one, gives in the limit the linear Fokker-Planck equation. Hence, formation of tails is expected in case of dissipation of energy, as well in case of production of energy, but only when the mixing parameter $p > 1$.

Remark 2.4. *In addition to the conservative case, a second one deserves to be mentioned. If $p = 1 - q$, the kinetic models is nothing but the model for granular dissipative collisions introduced and studied in*[1,3,25] *as a one-dimensional caricature of the Maxwell-Boltzmann equation (5, 6). In this case $\lambda = -1$, and the stationary state is*

$$M_1(v) = \frac{2}{\pi} \left(\frac{1}{1+v^2} \right)^2. \tag{25}$$

This solution solves the kinetic Eq. (10), for any value of the parameter $q < 1/2$.

Remark 2.5. *The asymptotic procedure considered in this section is the analogue of the so-called grazing collision limit of the Boltzmann equation,*[36,37] *which relies* *in concentrating the rate functions on collisions which are grazing, so leaving the collisional velocities unchanged. It is well-known that in this (conservative) case, while the Boltzmann equation changes into the Landau-Fokker-Planck equation, the stationary distribution remains of Maxwellian type.*

2.2. Pareto Tails in Kinetic Models of Economy

In this section we show how to extend the asymptotic analysis of the previous section to the case in which the kinetic model describes the time evolution of a density $f(v,t)$, which now denotes the distribution of wealth $v \in I\!R_+$ among economic agents at time $t \geq 0$. The collision (1) represents now a trade between individuals. For a deep insight into the matter, we address the interested reader to,[16,17,20,29,30] and to the references therein. With the convention $f(v,t) = 0$ if $v < 0$, the kinetic model reads.[13,26]

$$\frac{\partial f(v)}{\partial t} = \int_{I\!R_+} \left(\frac{1}{J} f(v_*) f(w_*) - f(v) f(w) \right) dw \tag{26}$$

where $(v^*, w^*) \in I\!R_+$ are the post-trade wealths generated by the couple (v, w) after the interaction, along the rule (1). As before, the jacobian $J = p^2 - q^2$. Since the v-variable takes values in $I\!R_+$, the collision rules (1) lead to a remarkable difference with respect to the case treated in the previous section. The pair (v_*, w_*) of pre-collision variables that generate the pair (v, w) is given by

$$v_* = \frac{pv - qw}{J}, \quad w_* = \frac{pw - qv}{J}.$$

While in the former case this pair is always admissible ($v_*, w_* \in I\!R$), in the latter we have to discard all pairs of pre-collision variables for which $v_* < 0$ or $w_* < 0$. This shows that, for any given $v \in I\!R_+$, the product $f(v_*)f(w_*)$ in (26) is different from zero only on the set $\mathcal{B} = \{(q/p)v < w < (p/q)v\}$. This implies in other words that, if we fix the wealth $v \in I\!R_+$ as outcome of a single trade, the other outcome w can only lie on the subset $\mathcal{B}$.

A great simplification is obtained writing Eq. (26) in weak form, where the presence of pre-collision wealths is avoided,

$$\frac{d}{dt} \int_{I\!R_+} \phi(v) f(v,t)\, dv = \int_{I\!R_+^2} f(v,t) f(w,t) (\phi(v^*) - \phi(v))\, dv\, dw. \tag{27}$$

Remark 2.6. *The role of the energy is now played by the mean* $m(t) = \int v f(v,t)\, dv$. *Note however that one can think to Eq. (26) as the analogous of the isotropic form of a hard-sphere Boltzmann equation for a density function* $f(v', t)$, $v' \in I\!R$ *written with respect to energy variable* $v = (v')^2/2$. *In this sense* *it is again the non conservation of the energy that will originate the power law tails.*

To look for self-similarity we scale our solution according to

$$g(v,t) = m(t) f(m(t)v, t), \tag{28}$$

which implies that $\int vg(v,t) = 1$ for all $t > 0$. Moreover $g = g(v,t)$ satisfies the equation

$$\frac{d}{dt}\int_{I\!R_+} \phi(v) g(v,t)\,dv - (p+q-1)\int_{I\!R_+} \phi(v)\frac{\partial}{\partial v}(vg)\,dv$$
$$= \int_{I\!R_+^2} g(v)g(w)(\phi(v^*) - \phi(v))\,dv\,dw. \tag{29}$$

Performing the same computations of the previous section, and *mutatis mutandis* we conclude that $g(v,t)$ satisfies

$$\frac{d}{dt}\int_{I\!R_+} \phi(v)g(v,t)\,dv + q\int_{I\!R} \phi'(v)(v-1)g(v)\,dv = \frac{1}{2}\int_{I\!R} g(v)$$
$$\times ((p-1)^2 v^2 + q^2 w^2 + 2(p-1)qvw)\phi''(v)\,dv + R(p,q). \tag{30}$$

The form of the remainder $R(p,q)$ is analogous to that of (13). It is clear that the correct scaling for small values of the parameter q is now

$$\tau = qt, \quad h(v,\tau) = g(v,t), \tag{31}$$

which implies that $h(v,\tau)$ satisfies the equation

$$\frac{d}{d\tau}\int_{I\!R_+} \phi(v)h(v,\tau)\,dv + \int_{I\!R} \phi'(v)(v-1)h(v)\,dv$$
$$= \frac{1}{2}\int_{I\!R} h(v)\frac{(p-1)^2}{q} v^2\phi''(v)dv + R_1(p,q), \tag{32}$$

where the remainder R_1 is given by

$$R_1(p,q) = \frac{1}{2}\int_{I\!R_+} (qw^2 + 2(p-1)vw)\phi''(v)\,dv + \frac{1}{q}R(p,q).$$

Let us consider a parameter $p = p(q)$ such that, for a given constant $\lambda > 0$

$$\lim_{q\to 0} \frac{(p(q)-1)^2}{q} = \lambda. \tag{33}$$

Then, Eq. (32) is well-approximated by the equation (in weak form)

$$\frac{d}{d\tau}\int_{I\!R} \phi(v)h(v,\tau)\,dv + \int_{I\!R} \phi'(v)(v-1)h(v)\,dv = \frac{\lambda}{2}\int_{I\!R} h(v)v^2\phi''(v)\,dv. \tag{34}$$

Equation (34) is nothing but the weak form of the Fokker-Planck equation

$$\frac{\partial h}{\partial \tau} = \frac{\lambda}{2}\frac{\partial^2}{\partial v^2}(v^2 h) + \frac{\partial}{\partial v}[(v-1)h], \tag{35}$$

which admits a unique stationary state of unit mass, given by the Γ-distribution[9,13]

$$M_\lambda(v) = \frac{(\mu-1)^\mu}{\Gamma(\mu)}\frac{\exp\left(-\frac{\mu-1}{v}\right)}{v^{1+\mu}} \tag{36}$$

where

$$\mu = 1 + \frac{2}{\lambda} > 1.$$

This stationary distribution exhibits a Pareto power law tail for large v's.

Note that this equation is essentially the same Fokker-Planck equation derived from a Lotka-Volterra interaction in[9,23,33]

Remark 2.7. *The formal analysis shows that the Fokker-Planck Equation (34) follows from the kinetic model independently of the sign of the quantity $p+q-1$, which can produce exponential growth of wealth (when positive), or exponential dissipation of wealth (when negative). Hence, Pareto tails are produced in both situations, as soon as the compatibility condition (33) holds. As discussed in Remark 2.3., condition (33) is always admissible if $p+q-1<0$, while one has to require $p>1$ if $p+q-1>0$. This is quite remarkable since it shows that this uneven distribution of money which characterizes most western economies may not only be produced as the effect of a growing economy but also under critical economical circumstances.*

Remark 2.8. *The model studied in*[32] *corresponds to the choice $p = 1-q+\epsilon$, with $\epsilon > 0$. This interaction implies exponential growth of wealth, and convergence of the solution to the Fokker-Planck equation if $\epsilon = \epsilon(q)$ satisfies*

$$\lim_{q\to 0}\frac{\epsilon^2(q)}{q} = \lambda.$$

Since the same limit equation is derived within the choice $p = 1-q-\epsilon$, we are free to choose ϵ negative. The particular choice

$$\epsilon = -2\sqrt{q} + 2q,$$

which implies $\mu = 3/2$ and thus $\lambda = 4$, leads to the stationary state [32]

$$M_4(v) = \frac{1}{\sqrt{2\pi}}\frac{\exp\left(-\frac{1}{2v}\right)}{v^{5/2}}, \tag{37}$$

which solves the kinetic Eq. (29) for all values of the scaling parameter $q < 1/4$.

3. THE FOURIER TRANSFORM OF THE KINETIC EQUATIONS

The formal results of Sec. 2.1 and 2.2 suggest that, at least in the limit $p \to 1$ and $q \to 0$, the large-time behavior of the solution to the kinetic model (9) is characterized by the presence of overpopulated tails. In what follows, we will justify rigorously this behavior, at least for a certain domain of the mixing parameters p and q. We start our analysis with a detailed study of the Boltzmann model (2).

The initial value problem for this model can be easily studied using its weak form (4). Let $\mathcal{M}_0$ the space of all probability measures in $I\!R_+$ and by

$$\mathcal{M}_\alpha = \left\{\mu \in \mathcal{M}_0 : \int_{I\!R} |v|^\alpha \mu(dv) < +\infty, \alpha > 0\right\}, \tag{38}$$

the space of all Borel probability measures of finite momentum of order α, equipped with the topology of the weak convergence of the measures.

By a weak solution of the initial value problem for Eq. (2), corresponding to the initial probability density $f_0(w) \in \mathcal{M}_\alpha, \alpha > 2$ we shall mean any probability density $f \in C^1(I\!R, \mathcal{M}_\alpha)$ satisfying the weak form of the equation

$$\frac{d}{dt}\int_{I\!R} \phi(v) f(v,t)\,dv = \int_{I\!R^2} f(v) f(w)(\phi(v^*) - \phi(v))\,dv\,dw, \tag{39}$$

for $t > 0$ and all smooth functions ϕ, and such that for all ϕ

$$\lim_{t\to 0}\int_{I\!R} \phi(v) f(v,t)\,dv = \int_{I\!R} \phi(v) f_0(v)\,dv. \tag{40}$$

In the rest of this section, we shall study the weak form of Eq. (2), with the normalization conditions (3). It is equivalent to use the Fourier transform of the equation (4):

$$\frac{\partial \hat{f}(\xi,t)}{\partial t} = \hat{Q}(\hat{f}, \hat{f})(\xi,t), \tag{41}$$

where $\hat{f}(\xi,t)$ is the Fourier transform of $f(x,t)$,

$$\hat{f}(\xi,t) = \int_{I\!R} e^{-i\xi v} f(v,t)\,dv,$$

and

$$\hat{Q}(\hat{f}, \hat{f})(\xi) = \hat{f}(p\xi)\hat{f}(q\xi) - \hat{f}(\xi)\hat{f}(0). \tag{42}$$

The initial conditions (3) turn into

$$\hat{f}(0) = 1,\ \hat{f}'(0) = 0,\ \hat{f}''(0) = -1,$$

$\hat{f} \in C^2(I\!R)$. Hence Eq. (41) can be rewritten as

$$\frac{\partial \hat{f}(\xi, t)}{\partial t} + \hat{f}(\xi, t) = \hat{f}(p\xi)\hat{f}(q\xi). \tag{43}$$

Equation (43) is a special case of equation (4.8) considered by Bobylev and Cercignani in [6]. Consequently, most of their conclusions applies to the present situation as well. The main difference here is that the mixing parameters p and q are allowed to assume values bigger than 1.

We introduce a metric on $\mathcal{M}_p$ by

$$d_s(f, g) = \sup_{\xi \in I\!R} \frac{|\hat{f}(\xi) - \hat{g}(\xi)|}{|\xi|^s} \tag{44}$$

Let us write $s = m + \alpha$, where m is an integer and $0 \leq \alpha < 1$. In order that $d_s(F, G)$ be finite, it suffices that F and G have the same moments up to order m.

The norm (44) has been introduced in [18] to investigate the trend to equilibrium of the solutions to the Boltzmann equation for Maxwell molecules. There, the case $s = 2 + \alpha$, $\alpha > 0$, was considered. Further applications of d_s can be found in.(10,11,19,35)

3.1. Uniqueness and Asymptotic Behavior

We will now study in details the asymptotic behavior of the scaled function $g(v, t)$. As briefly discussed before, a related analysis has been performed in the framework of the study of self-similar profiles for the Boltzmann equation for Maxwell molecules in.(6,7) Likewise, the role of the Fourier distance in the asymptotic study of nonconservative kinetic equations has been evidenced in.(31) Consequently, part of the results presented here fall into the results of,(6,31) and could be skipped. Nevertheless, for the sake of completeness, we will discuss the point in an exhaustive way.

The existence of a solution to Eq. (2) can be seen easily using the same methods available for the elastic Kac model. In particular, a solution can be expressed as a Wild sum.(4,10) In order to prove uniqueness, we use the method first introduced in.(18) Let f_1 and f_2 be two solutions of the Boltzmann equation (2), corresponding to initial values $f_{1,0}$ and $f_{2,0}$ satisfying conditions (3), and $\hat{f}_1$, $\hat{f}_2$ their Fourier transforms. Given any positive constant s, with $2 \leq s \leq 3$, let us suppose in addition that $d_s(f_{1,0}, f_{2,0})$ is bounded. Then, it holds

$$\frac{\partial}{\partial t} \frac{(\hat{f}_1 - \hat{f}_2)}{|\xi|^s} + \frac{\hat{f}_1(\xi) - \hat{f}_2(\xi)}{|\xi|^s} = \frac{\hat{f}_1(p\xi)\hat{f}_1(q\xi) - \hat{f}_2(p\xi)\hat{f}_2(q\xi)}{|\xi|^s}. \tag{45}$$

Now, since $|\hat{f}_1(\xi)| \leq 1$ ($|\hat{f}_2(\xi)| \leq 1$), we obtain

$$\left|\frac{\hat{f}_1(p\xi)\hat{f}_1(q\xi) - \hat{f}_2(p\xi)\hat{f}_2(q\xi)}{|\xi|^s}\right| \leq |\hat{f}_1(p\xi)| \left|\frac{\hat{f}_1(q\xi) - \hat{f}_2(q\xi)}{|q\xi|^s}\right| q^s \tag{46}$$

$$+ |\hat{f}_2(q\xi)| \left|\frac{\hat{f}_1(p\xi) - \hat{f}_2(p\xi)}{|p\xi|^s}\right| p^s \leq \sup \left|\frac{\hat{f}_1 - \hat{f}_2}{|\xi|^s}\right| (p^s + q^s).$$

We set

$$h(t, \xi) = \frac{\hat{f}_1(\xi) - \hat{f}_2(\xi)}{|\xi|^s}.$$

The preceding computation shows that

$$\left|\frac{\partial h}{\partial t} + h\right| \leq (p^s + q^s)\|h\|_\infty. \tag{47}$$

Gronwall's lemma proves at once that

$$\|h(t)\|_\infty \leq \exp\{(p^s + q^s - 1)t\}\|h_0\|_\infty.$$

We have

Theorem 3.1. *Let $f_1(t)$ and $f_2(t)$ be two solutions of the Boltzmann equation (2), corresponding to initial values $f_{1,0}$ and $f_{2,0}$ satisfying conditions (3). Then, if for some $2 \leq s \leq 3$, $d_s(f_{1,0}, f_{2,0})$ is bounded, for all times $t \geq 0$,*

$$d_s(f_1(t), f_2(t)) \leq \exp\{(p^s + q^s - 1)t\}d_s(f_{1,0}, f_{2,0}). \tag{48}$$

In particular, let f_0 be a nonnegative density satisfying conditions (3). Then, there exists a unique weak solution $f(t)$ of the Boltzmann equation, such that $f(0) = f_0$. In case $p^s + q^s - 1 < 0$ the distance d_s is contracting exponentially in time.

Let us remark that, given a constant $a > 0$,

$$\sup_{\xi \in I\!R} \frac{|\hat{f}_1(a\xi) - \hat{f}_2(a\xi)|}{|\xi|^s} = a^s \sup_{\xi \in I\!R} \frac{|\hat{f}_1(a\xi) - \hat{f}_2(a\xi)|}{|a\xi|^s} = a^s d_s(f_1, f_2). \tag{49}$$

Hence, if $g(t)$ represents the solution $f(t)$ scaled by its energy like in (8),

$$\hat{g}(\xi) = \hat{f}\left(\frac{\xi}{\sqrt{E(t)}}\right),$$

and from (49) we obtain the bound

$$d_s(g_1(t), g_2(t)) = \sup_{\xi \in I\!R} \frac{|\hat{g}_1(\xi, t) - \hat{g}_2(\xi, t)|}{|\xi|^s} = \left(\frac{1}{\sqrt{E(t)}}\right)^s d_s(f_1(t), f_2(t)). \tag{50}$$

Using (48), we finally conclude that, if $g_1(t)$ and $g_2(t)$ are two solutions of the scaled Boltzmann equation (9), corresponding to initial values $f_{1,0}$ and $f_{2,0}$ satisfying conditions (3), then, if $2 \le s \le 3$, for all times $t \ge 0$,

$$d_s(g_1(t), g_2(t)) \le \exp\left\{\left[(p^s + q^s - 1) - \frac{s}{2}(p^2 + q^2 - 1)\right] t\right\} d_s(f_{1,0}, f_{2,0}). \tag{51}$$

Let us define, for $\delta \ge 0$,

$$\mathcal{S}_{p,q}(\delta) = p^{2+\delta} + q^{2+\delta} - 1 - \frac{2+\delta}{2}(p^2 + q^2 - 1). \tag{52}$$

Then, the sign of $\mathcal{S}_{p,q}$ determines the asymptotic behavior of the distance $d_s(g_1(t), g_2(t))$. In particular, if there exists an interval $0 < \delta < \bar{\delta}$ in which $\mathcal{S}_{p,q}(\delta) < 0$, we can conclude that $d_{2+\delta}(g_1(t), g_2(t))$ converges exponentially to zero. Note that, by construction, $\mathcal{S}_{p,q}(0) = 0$, and thus $\min_\delta\{\mathcal{S}_{p,q}\} \le 0$. The function (52) was first considered by Bobylev and Cercignani in [6]. The sign of $\mathcal{S}_{p,q}$, however was studied mainly for $p = 1 - q$, namely the case of the dissipative Boltzmann equation. In Figure 1 a numerical evaluation of the region where the minimum of the function $\mathcal{S}_{p,q}$ is negative for $p, q \in [0, 2]$ is reported.

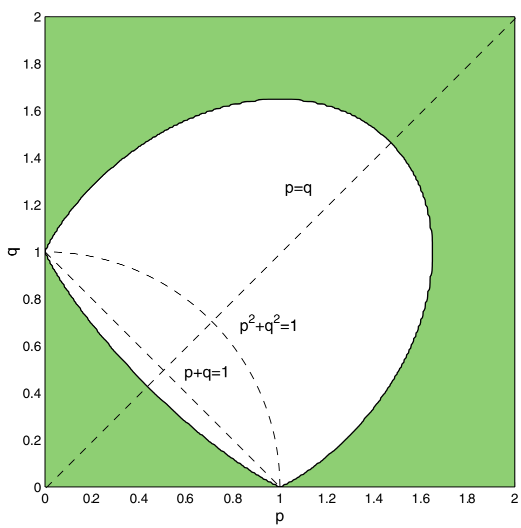

Fig. 1. The white domain represents the region where the minimum of the function $\mathcal{S}_{p,q}$ is negative for $p, q \in [0, 2]$.

Remark 3.2. *The behavior of $\mathcal{S}_{p,q}(\delta)$ when $p^2 + q^2 = 1$ is clear. In this case in fact, both p and q are less than 1, which implies*

$$\mathcal{S}_{p,q}(\delta) = p^{2+\delta} + q^{2+\delta} - 1 < 0,$$

for all $\delta > 0$. We can draw the same conclusion when $p^2 + q^2 > 1$, while both $p < 1$ and $q < 1$.

Consider now the case $p^2 + q^2 > 1$, with $p > 1$. In this case, while $-\frac{2+\delta}{2}(p^2 + q^2 - 1)$ decreases linearly, $p^{2+\delta}$ increases exponentially, and the sign of $\mathcal{S}_{p,q}(\delta)$ becomes positive for large values of δ.

If finally $p^2 + q^2 < 1$, the sign of $\mathcal{S}_{p,q}(\delta)$ for large values of δ is positive, since, while $p^{2+\delta} + q^{2+\delta} - 1 < 0$,

$$-\frac{2+\delta}{2}\left(p^2 + q^2 - 1\right) \geq 1$$

for

$$\delta \geq \frac{2(p^2 + q^2)}{1 - (p^2 + q^2)}.$$

The previous remark indicates that in the general case one can at best hope that $\mathcal{S}_{p,q}(\delta)$ is negative in an interval $(0, \bar{\delta})$. To show that this is really the case, one has to investigate carefully the behavior of $\mathcal{S}_{p,q}(\delta)$ in a neighborhood of zero. Since the function $\mathcal{S}_{p,q}(\delta)$ is convex for $\delta \geq 0$,

$$\frac{d^2\mathcal{S}_{p,q}(\delta)}{d\delta^2} = p^{2+\delta}(\log p)^2 + q^{2+\delta}(\log q)^2 > 0,$$

and $\mathcal{S}_{p,q}(0) = 0$, in all cases where $\mathcal{S}_{p,q}(\delta)$ is positive for large values of δ, a sufficient condition for $\mathcal{S}_{p,q}(\delta)$ be negative in some interval $0 < \delta < \bar{\delta}$ is that

$$\left.\frac{d\mathcal{S}_{p,q}(\delta)}{d\delta}\right|_{\delta=0} = p^2 \log p + q^2 \log q - \frac{1}{2}(p^2 + q^2 - 1) < 0.$$

Let us discuss before the case $p^2 + q^2 < 1$. Given $\lambda > 0$, we introduce a dependence between p and q by setting $p = 1 - \lambda q$. Since $p > q$, this relationship is possible only if $q < 1/(1 + \lambda)$. Moreover $p^2 + q^2 < 1$ requires $q < (2\lambda)/(1 + \lambda^2)$. Using this, it is immediate to show that there is an interval $0 \leq q \leq \bar{q}$ in which $\mathcal{S}'_{p,q}(0) < 0$. We have

$$\left.\frac{d\mathcal{S}_{p,q}(\delta)}{d\delta}\right|_{\delta=0} = G(q) = (1 - \lambda q)^2 \log(1 - \lambda q) + q^2 \log q$$
$$-\frac{1}{2}\left((1 - \lambda q)^2 + q^2 - 1\right).$$

Clearly, $G(0) = 0$. Moreover

$$G'(q) = 2q \log q - 2\lambda(1 - \lambda q) \log(1 - \lambda q),$$

and

$$G''(q) = 2(1 + \log q) + 2\lambda^2 (1 + \log(1 - \lambda q)).$$

Now $G''(q) < 0$ in some interval $(0, q_1)$, which implies that $G'(q)$ is decreasing in the same interval. But, since $G'(0) = 0$, $G'(q) < 0$ in the interval $(0, \bar{q})$, where $\bar{q}$ solves

$$2\bar{q} \log \bar{q} - 2\lambda(1 - \lambda\bar{q}) \log(1 - \lambda\bar{q}) = 0$$

Consequently, $G(q) < 0$ at least in the same interval.

Let us now treat the case $p^2 + q^2 > 1$, with $p > 1$. Let us set $p = 1 + \lambda q$. We have

$$\left.\frac{d\mathcal{S}_{p,q}(\delta)}{d\delta}\right|_{\delta=0} = G(q) = (1 + \lambda q)^2 \log(1 + \lambda q) + q^2 \log q - \frac{1}{2}\left((1 + \lambda q)^2 + q^2 - 1\right).$$

In this case

$$G'(q) = 2q \log q + 2\lambda(1 + \lambda q) \log(1 + \lambda q),$$

and

$$G''(q) = 2(1 + \log q) + 2\lambda^2 (1 + \log(1 + \lambda q)).$$

As before, $G''(q) < 0$ in some interval $(0, q_2)$, which implies that $G'(q)$ is decreasing in the same interval. But, since $G'(0) = 0$, $G'(q) < 0$ in the interval $(0, \bar{q})$, where $\bar{q}$ now solves

$$2\bar{q} \log \bar{q} + 2\lambda(1 + \lambda\bar{q}) \log(1 + \lambda\bar{q}) = 0$$

Consequently, $G(q) < 0$ at least in the same interval.

We proved

Lemma 3.3. *Let $\mathcal{S}_{p,q}(\delta), \delta > 0$ be the function defined by (52). Given a constant $\lambda > 0$, if $p^2 + q^2 < 1$, let us define $p = 1 - \lambda q$. Then, provided $q < \min\{1/(1 + \lambda), (2\lambda)/(1 + \lambda^2)\}$ there exists an interval $I_- = (0, \bar{\delta}_-(q))$ such that $\mathcal{S}_{p,q}(\delta) < 0$ for $\delta \in I_-$. If $p^2 + q^2 > 1$, and $p = 1 + \lambda q$ there exists an interval $I_+ = (0, \bar{\delta}_+(q))$ such that $\mathcal{S}_{p,q}(\delta) < 0$ for $\delta \in I_+$. In the remaining cases, namely when $p^2 + q^2 = 1$ or $p^2 + q^2 > 1$ but $p < 1$, $\mathcal{S}_{p,q}(\delta) < 0$ for all $\delta > 0$.*

Lemma 3.3 has important consequences both in the behavior of the solution to the Boltzmann equation (9), and in the limit procedure introduced in Sections 2.1 and 2.2. The main consequence of the lemma is contained into the following.

Theorem 3.4. *Let $g_1(t)$ and $g_2(t)$ be two solutions of the Boltzmann equation (9), corresponding to initial values $f_{1,0}$ and $f_{2,0}$ satisfying conditions (3). Then, there exists a constant $\bar{\delta} > 0$ such that, if $2 < s < 2 + \bar{\delta}$, for all times $t \geq 0$,*

$$d_s(g_1(t), g_2(t)) \leq \exp\{-C_s t\}\, d_s(f_{1,0}, f_{2,0}). \tag{53}$$

The constant $C_s = -\mathcal{S}_{p,q}(s-2)$ is strictly positive, and the distance d_s is contracting exponentially in time.

3.2. Convergence to Self-Similarity

By means of the estimates of Sec. 3.1, we will now discuss the evolution of moments for the solution to equation (9). By construction, the second moment of $g(v,t)$ is constant in time, and equal to 1 thanks to the normalization conditions (3). We can use the computations leading to the Fokker-Planck equation (23), choosing $\phi(v) = |v|^{2+\delta}$, where for the moment the positive constant $\delta \leq 1$. Suppose that the initial density $g_0(v) = f_0(v)$ is such that

$$\int_{I\!R} |v|^{2+\delta} g_0(v)\, dv = m_\delta < \infty. \tag{54}$$

Then, since the contribution due to the term $\frac{\partial}{\partial v}(vg(v))$ can be evaluated integrating by parts,

$$\int_{I\!R} |v|^{2+\delta} \frac{\partial}{\partial v}(vg(v))\, dv = -(2+\delta)\int_{I\!R} |v|^{2+\delta} g(v,t)\, dv,$$

we obtain

$$\frac{d}{dt}\int_{I\!R} |v|^{2+\delta} g(v,t)\, dv + (2+\delta)\frac{p^2+q^2-1}{2}\int_{I\!R} |v|^{2+\delta} g(v,t)\, dv \tag{55}$$
$$= \int_{I\!R^2} dv\, dw (|pv+qw|^{2+\delta} - |v|^{2+\delta}) g(v) g(w)\,.$$

Let us recover a suitable upper bound for the last integral in (55). Given any two constants a, b, and $0 < \delta \leq 1$ the following inequality holds

$$(|a|+|b|)^\delta \leq |a|^\delta + |b|^\delta. \tag{56}$$

Hence, choosing $a = p|v|$ and $b = q|w|$,

$$|pv+qw|^{2+\delta} \leq (pv+qw)^2 (p^\delta |v|^\delta + q^\delta |w|^\delta).$$

Substituting into the right-hand side of (55), recalling that the mean value of g is equal to zero, and the second moment of g equal to one, gives

$$\int_{I\!R^2} |pv + qw|^{2+\delta} g(v)g(w)\, dv\, dw \le \int_{I\!R^2} (pv + qw)^2 \times (p^\delta |v|^d + q^\delta |w|^d) g(v)g(w)\, dv\, dw = (p^{2+\delta} + q^{2+\delta}) \times \int_{I\!R} |v|^{2+\delta} g(v)\, dv + (p^2 q^\delta + q^2 p^\delta) \int_{I\!R} |v|^\delta\, dv.$$

Grouping all these inequalities, and recalling the expression of $\mathcal{S}_{p,q}(\delta)$ given by (52) we obtain the differential inequality

$$\frac{d}{dt} \int_{I\!R} |v|^{2+\delta} g(v,t)\, dv \le \mathcal{S}_{p,q}(\delta) \int_{I\!R} |v|^{2+\delta} g(v,t)\, dv + B_{p,\delta}, \tag{57}$$

where, by Hölder inequality

$$B_{p,\delta} \le p^2 q^\delta + q^2 p^\delta. \tag{58}$$

By Lemma 3.3, for any $\delta < \bar{\delta}$, $\mathcal{S}_{p,q}(\delta) < 0$. In this case, inequality (57) gives an upper bound for the moment, that reads

$$\int_{I\!R} |v|^{2+\delta} g(v,t)\, dv \le m_\delta + \frac{B_{p,\delta}}{|\mathcal{S}_{p,q}(\delta)|} < \infty. \tag{59}$$

In the case $\bar{\delta} > 3$ we can easily iterate our procedure to obtain that any moment of order $2 + \delta$, with $\delta < \bar{\delta}$ which is bounded initially, remains bounded at any subsequent time. The only difference now is that the explicit expression of the bound is more and more involved.

If $\delta < \bar{\delta}$, we can immediately draw conclusions on the large-time convergence of class of probability densities $\{g(v,t)\}_{t>0}$, By virtue of Prokhorov theorem (cfr.[22]) the existence of a uniform bound on moments implies that this class is tight, so that any sequence $\{g(v,t_n)\}_{n>0}$ contains an infinite subsequence which converges weakly to some probability measure g_∞. Thanks to our bound on moments, provided $\delta < \bar{\delta}$, g_∞ possesses moments of order $2 + \delta$, for $0 < \delta < \bar{\delta}$.

It is now immediate to show that this limit is unique. To this aim, let us consider two initial densities $f_{0,1}(v)$ and $f_{0,2}(v)$ such that, for some $0 < \delta < \bar{\delta}$,

$$\int_{\mathcal{R}} |v|^{2+\delta} f_{0,1}(v)\, dv < +\infty, \qquad \int_{\mathcal{R}} |v|^{2+\delta} f_{0,2}(v)\, dv < +\infty.$$

Then, by Theorem 3.4, the distance $d_s(f_1(t), f_2(t))$ between the solutions converges exponentially to zero with respect to time, as soon as $2 < s < 2 + \bar{\delta}$. Let now $f_0(v)$ possess finite moments of order $2 + \delta$, with $0 < \delta < \bar{\delta}$. Thanks to our previous computations on moments, for any fixed time $T > 0$, the corresponding solution $f(v, T)$ has finite moments of order $2 + \delta$. Choosing $f_{0,1}(v) = f_0(v)$, and

$f_{0,2}(v) = f(v, T)$ shows that $d_s(f(t), f(t+T))$ converges exponentially to zero in time. It turns out that the d_s-distance between subsequences converges to zero as soon as $\mathcal{S}_{p,q}(s-2) < 0$.

We can now show that the limit function $g_\infty(v)$ is a stationary solution to (9). We know that if condition (54) holds, both the solution $g(v,t)$ to equation (9) and $g_\infty(v)$ have moments of order $2+\delta$, with $0 < \delta < \bar{\delta}$ uniformly bounded. Hence, for any $t \geq 0$, proceeding as in the proof of Theorem 3.1, we obtain

$$d_s\left(Q(g(t), g(t)), Q(g_\infty, g_\infty)\right) \leq (p^s + q^s + 1)\, d_s\left(g(t), g_\infty\right). \tag{60}$$

This implies the weak* convergence of $Q(g(t), g(t))$ towards $Q(g_\infty, g_\infty)$. In particular, due to the equivalence among different metrics which metricize the weak* convergence of measures,(18,35) if $C_0^1(\mathbb{R})$ denotes the set of compactly supported continuously differentiable functions, endowed with it natural norm $\|\cdot\|_1$, for all $\phi \in C_0^1(\mathbb{R})$,

$$\int_{I\!R} \phi(v) Q(g(t), g(t))(v)\, dv \to \int_{I\!R} \phi(v) Q(g_\infty, g_\infty)(v)\, dv. \tag{61}$$

On the other hand, for all $\phi \in C_0^1(\mathbb{R})$, integration by parts gives

$$\int_{I\!R} \phi(v) \frac{\partial}{\partial v}\left(v g(v,t)\right) dv = -\int_{I\!R} v\phi'(v) g(v,t)\, dv. \tag{62}$$

Since $|v\phi'(v)| \leq |v| \|\phi'\|_1$, and the second moment of $g(v,t)$ is equal to unity, the convergence of $d_s\left(g(t), g_\infty\right)$ to zero implies

$$\int_{I\!R} v\phi'(v) g(v,t)\, dv \to \int_{I\!R} v\phi'(v) g_\infty(v)\, dv. \tag{63}$$

Finally, for all $\phi \in C_0^1(\mathbb{R})$ it holds

$$\int_{I\!R} \phi(v) \left\{ \frac{\partial}{\partial v}\left(v g_\infty(v)\right) - Q(g_\infty, g_\infty)(v) \right\} dv = 0. \tag{64}$$

This shows that g_∞ is the unique stationary solution to (9). We have

Theorem 3.5. *Let $\delta > 0$ be such that $\mathcal{S}_{p,q}(\delta) < 0$, and let $g_\infty(v)$ be the unique stationary solution to equation (9). Let $g(v,t)$ be the weak solution of the Boltzmann equation (9), corresponding to the initial density f_0 satisfying*

$$\int |v|^{2+\delta} f_0(v)\, dv < \infty.$$

Then, $g(v,t)$ satisfies

$$\int |v|^{2+\delta} g(v,t)\, dv \leq c_\delta < \infty.$$

If $0 < \delta \leq 1$ *the constant* c_δ *is given by (59). Moreover,* $g(v,t)$ *converges exponentially fast in Fourier metric towards* $g_\infty(v)$*, and the following bound holds*

$$d_{2+\delta}(g(t), g_\infty) \leq d_{2+\delta}(f_0, g_\infty) \exp\{-|\mathcal{S}_{p,q}(\delta)|t\} \tag{65}$$

where $\mathcal{S}_{p,q}(\delta)$ *is given by (52).*

Depending of the values of the mixing parameters p and q, the stationary solution g_∞ can have overpopulated tails. We can easily check the presence of overpopulated tails by looking at the singular part of the Fourier transform [14]. Since the Fourier transform of g_∞ satisfies the equation

$$-\frac{p^2+q^2-1}{2}\xi\frac{\partial \hat{g}}{\partial \xi} + \hat{g}(\xi) = \hat{g}(p\xi)\hat{g}(q\xi), \tag{66}$$

we set

$$\hat{g}(\xi) = 1 - |\xi|^2 + A|\xi|^{2+\delta} + \cdots \tag{67}$$

which takes into account the fact that g_∞ satisfies conditions (3). The leading small ξ-behavior of the singular component will reflect an algebraic tail of the velocity distribution. Substitution of expression (67) into (66) shows that the coefficient of the power $|\xi|^{2+\delta}$ is $A\mathcal{S}_{p,q}(\delta)$. Thus, the term $A|\xi|^{2+\delta}$ can appear in the expansion of $\hat{g}(\xi)$ as soon as δ is such that $\mathcal{S}_{p,q}(\delta) = 0$, $\delta > 0$. In other words, tails in the stationary distributions are present in all cases in which there exists a $\delta = \bar{\delta} > 0$ such that $\mathcal{S}_{p,q}(\bar{\delta}) = 0$. Now the answer is contained into Lemma 3.3.

3.3. The Grazing Collision Asymptotics

The results of the previous section are at the basis of the rigorous derivation of the Fokker-Planck asymptotics formally derived in Secs. 2.1 and 2.2. Suppose that the initial density $g_0(v) = f_0(v)$ satisfies condition (54). Using a Taylor expansion, we obtain

$$\begin{aligned} |pv+qw|^{2+\delta} - |v|^{2+\delta} &= (2+\delta)|v|^\delta v((p-1)v+qw) \\ &\quad + \frac{1}{2}(1+\delta)|\tilde{v}|^\delta((p-1)v+qw)^2, \end{aligned} \tag{68}$$

where, for some $0 \leq \theta \leq 1$,

$$\tilde{v} = \theta(pv+qw) + (1-\theta)v.$$

Using this into equality (55), one has

$$\frac{d}{dt}\int_{I\!R}|v|^{2+\delta}g(v,t)\,dv + (2+\delta)\frac{p^2+q^2-1}{2}\int_{I\!R}|v|^{2+\delta}g(v,t)\,dv = (2+\delta)$$

$$\times \int_{I\!R^2} (|v|^\delta v((p-1)v + qw))g(v)g(w)\,dv\,dw + \frac{1}{2}(2+\delta)(1+\delta)$$
$$\times \int_{I\!R^2} |\tilde{v}|^\delta((p-1)v + qw)^2 g(v)g(w)\,dv\,dw. \tag{69}$$

Since the momentum of g is equal to zero, we can rewrite (69) as

$$\frac{d}{dt}\int_{I\!R} |v|^{2+\delta} g(v,t)\,dv + \frac{2+\delta}{2}[(p-1)^2 + q^2]\int_{I\!R} |v|^{2+\delta} g(v,t)\,dv \le$$
$$+\frac{1}{2}(2+\delta)(1+\delta)\int_{I\!R^2} |\tilde{v}|^\delta((p-1)v + qw)^2 g(v)g(w)\,dv\,dw. \tag{70}$$

Assuming $0 < \delta < 1$,

$$|\tilde{v}| \le (1+p)^\delta |v|^\delta + q^\delta |w|^\delta.$$

Hence, if $|p-1|/q = \lambda$, we obtain the bound

$$\frac{d}{dt}\int_{I\!R} |v|^{2+\delta} g(v,t)\,dv + \frac{2+\delta}{2} q^2 \left[1+\lambda^2\right]\int_{I\!R} |v|^{2+\delta} g(v,t)\,dv \le$$
$$+\frac{1}{2}(2+\delta)(1+\delta)q^2 \int_{I\!R^2} ((1+p)^\delta |v|^\delta + q^\delta |w|^\delta)(\lambda v + w)^2 g(v)g(w)\,dv\,dw,$$

or, what is the same,

$$\frac{d}{dt}\int_{I\!R} |v|^{2+\delta} g(v,t)\,dv \le q^2 C(\lambda, q)\int_{I\!R} |v|^{2+\delta} g(v,t)\,dv\,. \tag{71}$$

If we now use (31), it holds

$$\frac{d}{d\tau}\int_{I\!R} |v|^{2+\delta} h(v,\tau)\,dv \le C(\lambda, q)\int_{I\!R} |v|^{2+\delta} h(v,\tau)\,dv\,, \tag{72}$$

namely the uniform boundedness of the $(2+\delta)$-moment of $h(v,\tau)$ with respect to q, for any fixed time τ.

Consider now the remainder (13), which can be rewritten as

$$R(p,q) = \frac{q^2}{2}\int_{I\!R^2} \left(\frac{p-1}{q}v + w\right)^2 \left(\phi''(\tilde{v}) - \phi''(v)\right) h(v)h(w)\,dv\,dw. \tag{73}$$

We need the following

Definition 3.6. *Let $\mathcal{F}_s(I\!R)$, be the class of all real functions ϕ on $I\!R$ such that $\phi^{(m)}(v)$ is Hölder continuous of order δ,*

$$\|\phi^{(m)}\|_\delta = \sup_{v \ne w} \frac{|\phi^{(m)}(v) - \phi^{(m)}(w)|}{|v-w|^\delta} < \infty, \tag{74}$$

the integer m and the number $0 < \delta \leq 1$ are such that $m + \delta = s$, and $\phi^{(m)}$ denotes the m-th derivative of g.

If $\phi \in \mathcal{F}_s(I\!R)$, with $s = 2 + \delta$,

$$\left|\phi''(\tilde{v}) - \phi''(v)\right| \leq \|\phi''\|_\delta |\tilde{v} - v|^\delta \leq \|\phi''\|_\delta |(p-1)v + qw|^\delta. \tag{75}$$

In this case,

$$\begin{aligned} R(p,q) &\leq \frac{q^{2+\delta}}{2} \; \|\phi''\|_\delta \int_{I\!R^2} \left(\frac{p-1}{q} v + w\right)^{2+\delta} h(v)h(w)\,dv\,dw \\ &\leq \frac{q^{2+\delta}}{2} \; \|\phi''\|_\delta C_2(\lambda, q) \int_{I\!R^2} |v|^{2+\delta} h(v)\,dv. \end{aligned} \tag{76}$$

Thanks to the uniform bound on $(2+\delta)$-moment of $h(v, \tau)$, it follows that, for any fixed time $\tau > 0$,

$$\lim_{q \to 0} \frac{1}{q^2} R(p,q) = 0 \tag{77}$$

as soon as $\phi \in \mathcal{F}_s(I\!R)$, with $s = 2 + \delta$. This implies that the limit equation is the Fokker-Planck equation (19). We proved

Theorem 3.7. *Let the probability density $f_0 \in \mathcal{M}_\alpha$, where $\alpha = 2 + \delta$ for some $\delta > 0$, and let the mixing parameters satisfy*

$$\frac{(p-1)^2}{q^2} = \lambda^2,$$

for some constant λ fixed. Then, as $q \to 0$, for all $\phi \in \mathcal{F}_s(I\!R)$, with $s = 2 + \delta$ the weak solution to the Boltzmann equation (17) for the scaled density $h(v, \tau) = g(v, t)$, with $\tau = q^2 t$ converges, up to extraction of a subsequence, to a probability density $h(w, \tau)$. This density is a weak solution of the Fokker-Planck equation (19).

3.4. A Comparison of Tails

The result of Sec. 3.3 establishes a rigorous connection between the collisional kinetic equation (2) and the Fokker-Planck equation (19). The result of Lemma 3.3, coupled with the comment of Remark 2.3. then shows that there exists a link between tails of the stationary solution of Fokker-Plank and Boltzmann equations. In fact, one can choose $\lambda^2 > 0$ in Theorem 3.7 if and only if the mixing parameters p and q satisfy the conditions of the aforementioned Lemma 3.3. Since the reckoning of the size of the tails is immediate in the Fokker-Planck case, it would be important to know if one can extract from this knowledge information about size of the tails of the Boltzmann equation.

Since the size of tails in the Boltzmann equation is given by the positive root of the equation

$$\mathcal{S}_{p,q}(\delta) = 0,$$

where $\mathcal{S}_{p,q}$ is the function (52), we will try to extract information by comparing this root with the value of the parameter λ that characterizes the tails of the Fokker-Planck equation. If $p > 1$, using a Taylor expansion of $\mathcal{S}_{p,q}(\delta)$, with $p = 1 + \lambda q$, we obtain

$$\frac{\mathcal{S}_{p,q}(\delta)}{q^2} = \frac{2+\delta}{2}\left[(\lambda^2\delta - 1) + \frac{2}{2+\delta}\, q^{\delta} + \frac{(1+\delta)\delta}{3}\lambda^3 \frac{\bar{q}^3}{q^2}\right], \tag{78}$$

where $0 \le \bar{q} \le q$. This shows that, in the scaling of Theorem 3.7, the positive root $\delta^*(q)$ of $\mathcal{S}_{p,q}(\delta) = 0$ converges, as $q \to 0$ to the value $1/\lambda^2$, which characterizes the tails of the Fokker-Planck equation. When $\lambda > 0$, one can easily argue that $\delta^*(q) < 1/\lambda^2$. In this case, in fact,

$$\frac{\mathcal{S}_{p,q}(\delta)}{q^2} = \frac{2+\delta}{2}[\left(\lambda^2\delta - 1\right) + A], \tag{79}$$

where $A > 0$ if $q > 0$. Hence

$$\frac{\mathcal{S}_{p,q}(1/\lambda^2)}{q^2} = \frac{2+\delta}{2} A > 0, \tag{80}$$

that, by virtue of the convexity properties of $\mathcal{S}_{p,q}(\delta)$ implies $\delta^*(q) < 1/\lambda^2$.

A weaker information can be extracted when $p < 1$ while $p^2 + q^2 < 1$. In this case, writing $p = 1 - \lambda q$, $\lambda > 0$, we obtain

$$\frac{\mathcal{S}_{p,q}(\delta)}{q^2} = \frac{2+\delta}{2}\left[\left(\lambda^2\delta - 1\right) + \frac{2}{2+\delta}\, q^{\delta} - \frac{(1+\delta)\delta}{3}\lambda^3 \frac{\bar{q}^3}{q^2}\right], \tag{81}$$

where $0 \le \bar{q} \le q$. Let us set

$$q \le \frac{B\lambda}{1+\lambda^2}, \tag{82}$$

where $B \le 2$. In fact, when $p < 1$ Lemma 3.3 implies that there is formation of tails only when p and q are such that $p^2 + q^2 < 1$, which is equivalent to the condition

$$q < \frac{2\lambda}{1+\lambda^2}. \tag{83}$$

Hence, when q satisfies (82), from (81) we obtain the inequality

$$\frac{\mathcal{S}_{p,q}(\delta)}{q^2} \frac{2+\delta}{2}\left[\left(\lambda^2\delta - 1\right) - B\frac{(1+\delta)\delta}{3}\frac{\lambda^4}{1+\lambda^2}\right]. \tag{84}$$

Easy computations then show that, if $\delta = r\lambda^2$, with $0 < r < 1$, the right-hand side of (84) is nonnegative as soon as

$$3r(1-r)(1+\lambda^2)B(1+r\lambda^2).$$

Hence, the biggest value of B for which the right-hand side of (84) is nonnegative is attained when $r = 1/2$. In this case, $B = 3/4$, and $\delta^*(q) < 2/\lambda^2$. We can collect the previous analysis into the following

Lemma 3.8. *Let the mixing parameters satisfy*

$$\frac{(p-1)^2}{q^2} = \lambda^2,$$

for some constant λ fixed. Then, if $p > 1$ the positive root $\delta^*(q)$ of the equation $\mathcal{S}_{p,q}(\delta) = 0$, characterizing the tails of the Boltzmann equation, satisfies the bound $\delta^*(q) < 1/\lambda^2$. If $p < 1$, and at the same time q satisfies the bound (82) with $B = 3/4$, the positive root $\delta^*(q)$ of the equation $\mathcal{S}_{p,q}(\delta) = 0$, satisfies the bound $\delta^*(q) < 2/\lambda^2$.

We remark here that, in the case $p < 1$, setting $\delta = 1$ we obtain an exact formula for $\mathcal{S}_{p,q}(1)$,

$$\frac{\mathcal{S}_{p,q}(1)}{q^2} = \frac{3}{2}\left[\left(\lambda^2 - 1\right) + \frac{2}{3}q - \frac{2}{3}\lambda^3 q\right]. \tag{85}$$

Choosing $\lambda = 1$, we get $\mathcal{S}_{p,q}(1) = 0$. This case, that corresponds to the conservation of momentum in the Boltzmann equation has tails which are invariant with respect to q (see Remark 2.4.).

3.5. Kinetic Models of Economy

The analysis of Sect. 3, 3.1, 3.2 and 3.3 can be easily extended to equation (26) for the wealth distribution. We can in fact resort to the methods introduced for the kinetic equation on the whole real line simply setting

$$F(v,t) = f(v,t)I(v > 0), \qquad v \in I\!R, \tag{86}$$

where $I(A)$ is the indicator function of the set A. With this notation, equation (26) can be rewritten as equation (2),

$$\frac{\partial F(v)}{\partial t} = \int_{I\!R}\left(\frac{1}{J}F(v_*)F(w_*) - F(v)F(w)\right)dw. \tag{87}$$

Likewise, the weak form (27) reads

$$\frac{d}{dt}\int_{I\!R}\phi(v)F(v,t)\,dv = \int_{I\!R^2} F(v,t)F(w,t)(\phi(v^*) - \phi(v))\,dv\,dw. \tag{88}$$

We recall that the role of the energy is now supplied by the mean $m(t) = \int vF(v,t)\,dv$. To look for self-similarity we scale our solution according to

$$G(v,t) = m(t)F\left(m(t)v,t\right), \tag{89}$$

which implies that $\int vG(v,t) = 1$ for all $t \geq 0$. Hence, without loss of generality, if we fix the initial density to satisfy

$$\int_{I\!R} F_0(v)\,dv = 1\,; \qquad \int_{I\!R} vF_0(v)\,dv = 1\,, \tag{90}$$

the solution $G(v,t)$ satisfies (90). Then, the same computations of Sec. 3 show the following

Theorem 3.9. *Let $f_1(t)$ and $f_2(t)$ be two solutions of the Boltzmann equation (26), corresponding to initial values $f_{1,0}$ and $f_{2,0}$ satisfying conditions (90). Then, if for some $1 \leq s \leq 2$, $d_s(f_{1,0}, f_{2,0})$ is bounded, for all times $t \geq 0$,*

$$d_s(f_1(t), f_2(t)) \leq \exp\{(p^s + q^s - 1)t\}d_s(f_{1,0}, f_{2,0}). \tag{91}$$

In particular, let f_0 be a nonnegative density satisfying conditions (3). Then, there exists a unique weak solution $f(t)$ of the Boltzmann equation, such that $f(0) = f_0$. In case $p^s + q^s - 1 < 0$ the distance d_s is contracting exponentially in time.

Since by (89)

$$\hat{G}(\xi) = \hat{G}\left(\frac{\xi}{m(t)}\right),$$

from (49) we obtain the bound

$$d_s(g_1(t), g_2(t)) = \sup_{\xi \in I\!R} \frac{|\hat{G}_1(\xi,t) - \hat{G}_2(\xi,t)|}{|\xi|^s} = \left(\frac{1}{m(t)}\right)^s d_s(f_1(t), f_2(t)). \tag{92}$$

Using (91), we finally conclude that, if $g_1(t)$ and $g_2(t)$ are two solutions of the scaled Boltzmann equation (26), corresponding to initial values $f_{1,0}$ and $f_{2,0}$ satisfying conditions (90), Then, if $1 \leq s \leq 2$, for all times $t \geq 0$,

$$d_s(g_1(t), g_2(t)) \leq \exp\{[(p^s + q^s - 1) - s(p + q - 1)]t\Big\}\, d_s(f_{1,0}, f_{2,0}). \tag{93}$$

Let us define, for $\delta \geq 0$,

$$\mathcal{R}_{p,q}(\delta) = p^{1+\delta} + q^{1+\delta} - 1 - (1+\delta)(p + q - 1)\,. \tag{94}$$

Then, the sign of $\mathcal{R}_{p,q}$ now determines the asymptotic behavior of the distance $d_s(g_1(t), g_2(t))$. With few differences, the proof leading to Lemma 3.3 can be repeated, obtaining

Lemma 3.10. *Let $\mathcal{R}_{p,q}(\delta)$, $\delta \geq 0$ be the function defined by (94). Given a constant $\lambda > 0$, if $p + q < 1$, let us define $p = 1 - \lambda\sqrt{q}$. Then, provided $q < 1/\lambda^2$ there exists an interval $I_- = (0, \bar{\delta}_-(q))$ such that $\mathcal{R}_{p,q}(\delta) < 0$ for $\delta \in I_-$. If $p + q > 1$, and $p = 1 + \lambda\sqrt{q}$ there exists a interval $I_+ = (0, \bar{\delta}_+(q))$ such that $\mathcal{R}_{p,q}(\delta) < 0$ for $\delta \in I_+$. In the remaining cases, namely when $p + q = 1$ or $p + q > 1$ but $p < 1$, $\mathcal{R}_{p,q}(\delta) < 0$ for all $\delta > 0$.*

The main consequence of Lemma 3.10 is contained into the following.

Theorem 3.11. *Let $g_1(t)$ and $g_2(t)$ be two solutions of the Boltzmann equation (26), corresponding to initial values $f_{1,0}$ and $f_{2,0}$ satisfying conditions (90). Then, there exists a constant $\bar{\delta} > 0$ such that, if $1 < s < 1 + \bar{\delta}$, for all times $t \geq 0$,*

$$d_s(g_1(t), g_2(t)) \leq \exp\{-C_s t\}\, d_s(f_{1,0}, f_{2,0}). \tag{95}$$

The constant $C_s = -\mathcal{R}_{p,q}(s-1)$ is strictly positive, and the distance d_s is contracting exponentially in time.

Existence and uniqueness of the stationary solution to Eq. (29) follows along the same lines of Sec. 3.2. The main result is now contained into the following.

Theorem 3.12. *Let $\delta > 0$ be such that $\mathcal{R}_{p,q}(\delta) < 0$, and let $g_\infty(v)$ be the unique stationary solution to equation (29). Let $g(v, t)$ be the weak solution of the Boltzmann equation (29), corresponding to the initial density f_0 satisfying*

$$\int_{I\!R_+} |v|^{1+\delta} f_0(v)\, dv < \infty.$$

Then, $g(v, t)$ satisfies

$$\int_{I\!R_+} |v|^{1+\delta} g(v, t)\, dv \leq c_\delta < \infty,$$

for some constant c_δ depending only on p and q. Moreover, $g(v, t)$ converges exponentially fast in Fourier metric towards $g_\infty(v)$, and the following bound holds

$$d_{1+\delta}(g(t), g_\infty) \leq d_{1+\delta}(f_0, g_\infty) \exp\{-|\mathcal{R}_{p,q}(\delta)| t\} \tag{96}$$

where $\mathcal{R}_{p,q}(\delta)$ is given by (94).

Depending on the values of the mixing parameters p and q, the stationary solution g_∞ can have overpopulated tails. The Fourier transform of g_∞ satisfies the equation

$$-(p + q - 1)\xi \frac{\partial \hat{G}}{\partial \xi} + \hat{G}(\xi) = \hat{G}(p\xi)\hat{G}(q\xi). \tag{97}$$

We set

$$\hat{G}(\xi) = 1 - i\xi + A|\xi|^{1+\delta} + \dots \tag{98}$$

which takes into account the fact that g_∞ satisfies conditions (90). The leading small ξ-behavior of the singular component will reflect an algebraic tail of the velocity distribution. Substitution of expression (98) into (97) shows that the coefficient of the power $|\xi|^{1+\delta}$ is $A\mathcal{R}_{p,q}(\delta)$. Thus, the term $A|\xi|^{1+\delta}$ can appear in the expansion of $\hat{G}(\xi)$ as soon as δ is such that $\mathcal{R}_{p,q}(\delta) = 0$, $\delta > 0$. As before, tails in the stationary distributions are present in all cases in which there exists a $\delta = \bar{\delta} > 0$ such that $\mathcal{R}_{p,q}(\bar{\delta}) = 0$. Now the answer is contained into Lemma 3.10.

Last, one can justify rigorously the passage to the Fokker-Planck equation (35).

Theorem 3.13. *Let the probability density $f_0 \in \mathcal{M}_\alpha$, where $\alpha = 1 + \delta$ for some $\delta > 0$, and let the mixing parameters satisfy*

$$\frac{(p-1)^2}{q} = \lambda,$$

for some $\lambda > 0$ fixed. Then, as $q \to 0$, for all $\phi \in \mathcal{F}_s(I\!R)$, with $s = 1 + \delta$ the weak solution to the Boltzmann equation (32) for the scaled density $h(v, \tau) = g(v, t)$, with $\tau = qt$ converges, up to extraction of a subsequence, to a probability density $h(w, \tau)$. This density is a weak solution of the Fokker-Planck equation (35).

We finally remark that the discussion of Sec. 3.4, with minor modifications, can be adapted to establish connections between the size of the tails of the kinetic and Fokker-Planck models.

4. NUMERICAL EXAMPLES

In this paragraph, we shall compare the self-similar stationary results obtained by using Monte Carlo simulation of the kinetic model with the stationary state of the Fokker-Planck model. The method we adopted is based on Bird's time counter approach at each time step followed by a renormalization procedure according to the self-similar scaling used. We refer to (27) for more details on the use of Monte Carlo method for Boltzmann equations.

We used $N = 5000$ particles and perform several iterations until a stationary state is reached. The distribution is then averaged over the next 4000 iterations in order to reduce statistical fluctuations. Clearly, due to the slow convergence of the Monte Carlo method near the tails, some small fluctuations are still present for large velocities.

4.1. Gaussian Behavior

First we consider the case $\lambda = 0$ for which the steady state of the Fokker-Planck asymptotic is the Gaussian (24). We fix $p = 1$ so that for $q < 1/\sqrt{2}$ we expect Gaussian behavior also in the kinetic model. We report the results obtained for $q = 0.4$ and $q = 0.8$ in Figure 2.

4.2. Formation of Power Laws

Next we simulate the formation of power laws for positive λ. We take $p = 1.2$ and $q = .4$ which correspond to $\lambda = 0.5$. Keeping the same value of λ we then take $q = 0.1$ and $p = 1.05$. In Figure. 3 we plot the results showing convergence towards the Fokker-Planck behavior.

4.3. A Simple Growing Economy

We take the case of a growing economy for $p = 1 - q + 2\sqrt{q}$ thus corresponding to the limit Fokker-Planck steady state (36) with $\lambda = 2$ and $\mu = 2$. As prescribed from our theoretical analysis we observe that the equilibrium distribution converges toward the Fokker-Planck limit as q goes to 0, with λ fixed. The results are reported in Figures 4.

5. CONCLUSIONS

In this paper we studied the large-time behavior of a simple one-dimensional kinetic model of Maxwell type, in two situations, depending wether the velocity variable can take values on $\mathbb{R}$ or in $\mathbb{R}_+$, the former case describing nonconservative models of kinetic theory of rarefied gases, the latter elementary kinetic models of open economies. In both situations it has been shown that the lack of conservation laws leads to situations in which the self-similar solution has overpopulated tails. This is particularly important in the case of economy, where elementary explanations of the formation of Pareto tails can help to handle more complex models of society wealth distribution, where various other factors occur. It would be certainly interesting to extend a similar analysis to more realistic situations. Recently, a kinetic model including market returns has been introduced.(13) While for this model the asymptotic convergence to the Fokker-Planck limit can be obtained, the property of creation of overpopulated tails has been shown only by numerical simulation. In realistic models, in fact, there is a strong correlation among densities, due to the constraint of having non-negative wealths after trades, and this appears difficult to treat from a mathematical point of view. A further point deserves to be mentioned. Recent studies have shown that, while overpopulated tails seem to be generic feature of the non-conservative collision mechanism, in the kinetic theory

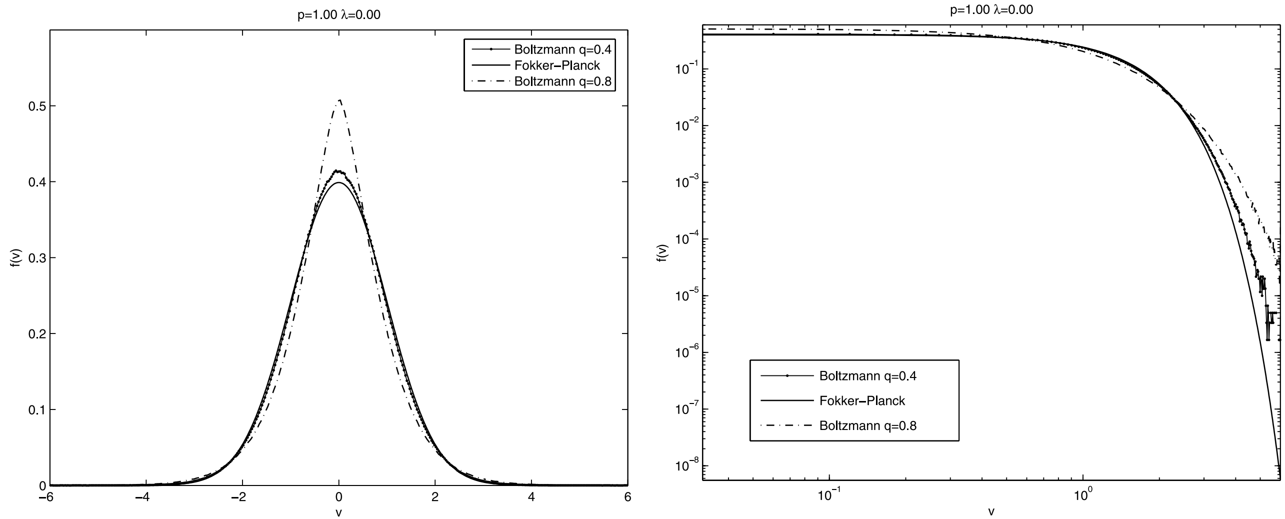

Fig. 2. Asymptotic behavior for $\lambda = 0$ of the Fokker-Planck model and the Boltzmann model with $p = 1$ and $q = 0.4, 0.8$. Figure on the right is in loglog-scale.

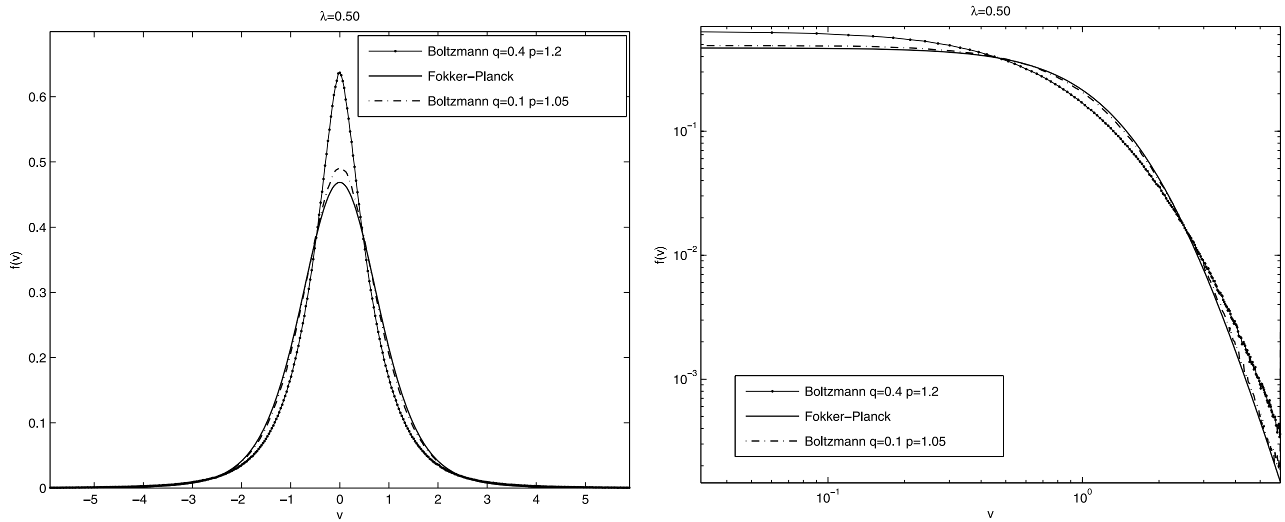

Fig. 3. Asymptotic behavior for $\lambda = 0.5$ of the Fokker-Planck model and the Boltzmann model for $p = 1.2$, $q = 0.4$ and $p = 1.05$, $q = 0.1$. Figure on the right is in loglog-scale.

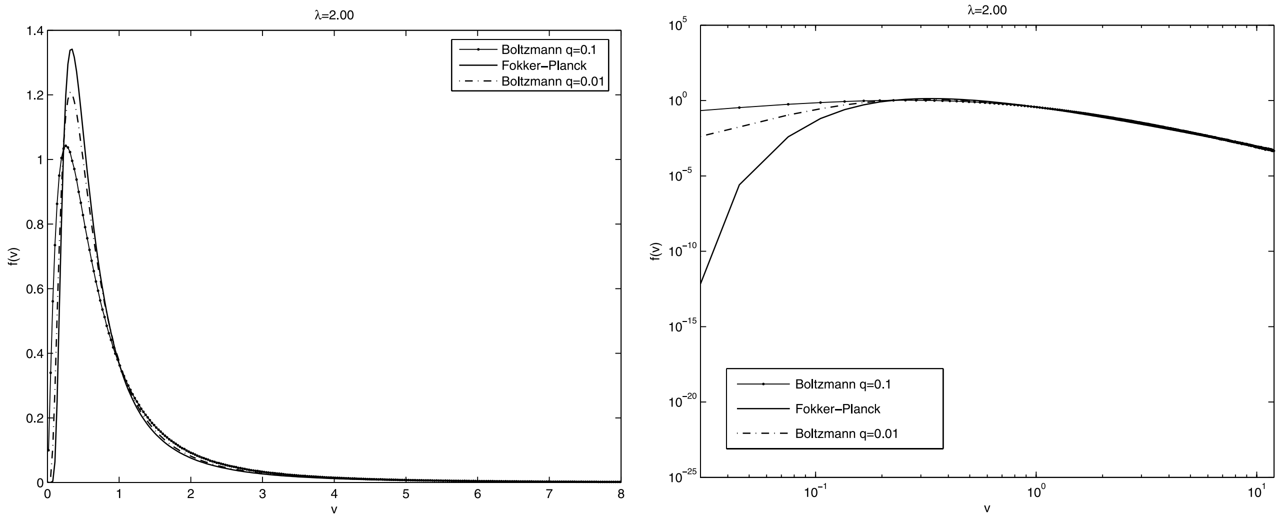

Fig. 4. Asymptotic behavior for $\lambda = 2$ of the Fokker-Planck model and the Boltzmann model for $p = 1 - q + 2\sqrt{q}$, $q = 0.1$ and $q = 0.01$. Figure on the right is in loglog-scale.

of the Boltzmann equation power-like tails only occur in the *borderline* case of Maxwell molecules interactions,[6,7,14,15] whereas in general collision dissipative processes have stretched exponential tail behaviors.[8] It could be conjectured that the corresponding phenomenon in general kinetic models of economy with wealth-depending collision frequency manifest a behavior in the form of a lognormal type distribution.

ACKNOWLEDGMENTS

The authors acknowledge support from the IHP network HYKE "Hyperbolic and Kinetic Equations: Asymptotics, Numerics, Applications" HPRN-CT-2002-00282 funded by the EC., and from the Italian MIUR, project "Mathematical Problems of Kinetic Theories."

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